
8.1 Turbulence Models For General-Purpose CFD

Turbulence models for general-purpose CFD must be *frame-invariant* – i.e. independent of any particular coordinate system – and hence must be expressed in tensor form. This rules out simpler models of boundary-layer type (e.g. mixing-length models).

Turbulent flows are computed either by solving the Reynolds-averaged Navier-Stokes equations with suitable models for turbulent fluxes or by computing the fluctuating quantities directly. The main approaches are summarised below.

Reynolds-Averaged Navier-Stokes (RANS) Models

- **Linear eddy-viscosity models (EVM)**
  – assume that the (deviatoric) turbulent stress is proportional to the mean strain;
  – use an eddy viscosity constructed from turbulence scalars (usually k + one other), determined by solving transport equations.

- **Non-linear eddy-viscosity models (NLEVM)**
  – assume that the turbulent stress is a non-linear function of mean strain and vorticity;
  – use coefficients constructed from turbulence scalars (usually k + one other), determined by solving transport equations;
  – mimic response of turbulence to certain important types of strain.

- **Differential stress models (DSM)**
  – aka Reynolds-stress transport models (RSTM) or second-order closure (SOC);
  – solve transport equations for all turbulent fluxes.

Computation of fluctuating quantities

- **Large-eddy simulation (LES)**
  – compute time-varying flow, but model sub-grid-scale motions.

- **Direct numerical simulation (DNS)**
  – no modelling; resolve the smallest scales of the flow.

8.2 Linear Eddy-Viscosity Models

8.2.1 General Form

Stress-strain constitutive relation:

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k\delta_{ij}, \qquad \mu_t = \rho\nu_t \tag{1}$$

The *eddy viscosity* μ_t is derived from turbulent quantities such as the turbulent kinetic energy k and dissipation rate ε. These quantities are themselves determined by solving scalar-transport equations (see below).

A typical shear stress and normal stress are given by

$$-\rho\overline{uv} = \mu_t\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

$$-\rho\overline{u^2} = 2\mu_t\frac{\partial U}{\partial x} - \frac{2}{3}\rho k$$

From these the other stress components are easily deduced by inspection/cyclic permutation.

<u>General Comments</u>

- μ is a physical property of the *fluid* and can be measured; μ_t is a hypothetical property of the *flow* and must be modelled.
- μ_t varies with position.
- At high Reynolds numbers, $\mu_t \gg \mu$ throughout much of the flow.

<u>Advantages</u>

- Easy to implement in viscous solvers.
- Extra viscosity aids stability.
- Some theoretical foundation in simple shear flows.

<u>Disadvantages</u>

- Little turbulence physics; in particular, *anisotropy* and *history* effects are neglected.
- Turbulent transport of momentum is determined by a single scalar μ_t, so at most one Reynolds stress ($-\rho\overline{uv}$) can be represented accurately; such models are questionable in complex flow.

Most eddy-viscosity models in general-purpose CFD codes are of the 2-equation type; (i.e. scalar-transport equations are solved for 2 turbulent scales). The commonest types are k-ε and k-ω models, for which specifications are given below.

8.2.2 *k*-ε Models

Eddy viscosity:

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \tag{2}$$

Scalar-transport equations (non-conservative form):

$$\begin{aligned}
\rho\frac{\mathrm{D}k}{\mathrm{D}t} &= \frac{\partial}{\partial x_i}(\Gamma^{(k)}\frac{\partial k}{\partial x_i}) &&+\rho(P^{(k)} &&-\varepsilon) \\
\rho\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} &= \frac{\partial}{\partial x_i}(\Gamma^{(\varepsilon)}\frac{\partial \varepsilon}{\partial x_i}) &&+\rho(C_{\varepsilon 1}P^{(k)} &&-C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k} \\
\textit{rate of change} & \quad \textit{diffusion} && \textit{production} && \textit{dissipation}
\end{aligned} \tag{3}$$

Diffusivities $\Gamma^{(k)}$ and $\Gamma^{(\varepsilon)}$ are related to the eddy viscosity via *Prandtl numbers* σ:

$$\Gamma^{(k)} = \mu + \frac{\mu_t}{\sigma^{(k)}}, \qquad \Gamma^{(\varepsilon)} = \mu + \frac{\mu_t}{\sigma^{(\varepsilon)}}$$

and the rate of production of turbulent kinetic energy (per unit mass) is

$$P^{(k)} \equiv -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \tag{4}$$

In the standard *k*-ε model (Launder and Spalding, 1974) the coefficients take the values

$$C_\mu = 0.09, \qquad C_{\varepsilon 1} = 1.92, \qquad C_{\varepsilon 2} = 1.44, \qquad \sigma^{(k)} = 1.0, \qquad \sigma^{(\varepsilon)} = 1.3 \tag{5}$$

Other important variants include RNG *k*-ε (Yakhot et al., 1992) and low-Re models such as Launder and Sharma (1974), Lam and Bremhorst (1981), and Lien and Leschziner (1993).

Modifications are employed in low-Re models to incorporate effects of molecular viscosity. Specifically, C_μ, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are multiplied by viscosity-dependent factors f_μ, f_1 and f_2 respectively, and an additional source term $S^{(\varepsilon)}$ may be required in the ε equation. Some models (notably Launder and Sharma, 1974) solve for the *homogeneous* dissipation rate $\tilde{\varepsilon}$ which vanishes at solid boundaries and is related to ε by

$$\varepsilon = \tilde{\varepsilon} + D, \qquad D = 2\nu(\nabla k^{1/2})^2 \tag{6}$$

This is consistent with the theoretical near-wall behaviour, $\varepsilon \sim 2\nu k / y^2$.

8.2.3 *k*-ω Models

ω (nominally equal to $\frac{\varepsilon}{C_\mu k}$) is sometimes known as the *specific dissipation rate* and has dimensions of 1/time, or frequency.

Eddy viscosity:

$$\nu_t = \frac{k}{\omega} \tag{7}$$

Scalar-transport equations:

$$\rho \frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial x_i}(\Gamma^{(k)} \frac{\partial k}{\partial x_i}) + \rho(P^{(k)} - \beta^* \omega k)$$
$$\rho \frac{\mathrm{D}\omega}{\mathrm{D}t} = \frac{\partial}{\partial x_i}(\Gamma^{(\omega)} \frac{\partial \omega}{\partial x_i}) + \rho(\frac{\alpha}{\nu_t} P^{(k)} - \beta \omega^2) \qquad (8)$$

The diffusivities of k and ω are related to the eddy-viscosity:

$$\Gamma^{(k)} = \mu + \frac{\mu_t}{\sigma^{(k)}}, \qquad \Gamma^{(\omega)} = \mu + \frac{\mu_t}{\sigma^{(\omega)}}$$

The original k-ω model was that of Wilcox (1988a) with coefficients taking the values

$$\beta^* = \frac{9}{100}, \quad \alpha = \frac{5}{9}, \quad \beta = \frac{3}{40}, \quad \sigma^{(k)} = 2.0, \quad \sigma^{(\omega)} = 2.0 \qquad (9)$$

The model was further developed by Wilcox (1998) in his book, with the coefficients becoming functions of the turbulent Reynolds number.

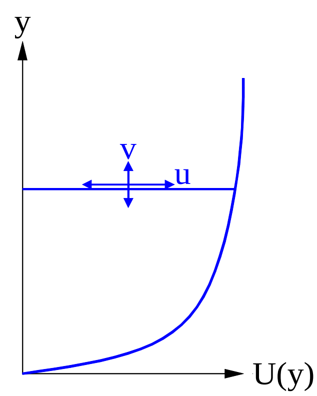
Menter (1994) devised a *shear-stress-transport* (SST) model. The model, which is expressed in k-ω form, blends the k-ω model (which is – allegedly – superior in the near-wall region), with the k-ε model (which is less sensitive to the level of turbulence in the free stream).

All models of k-ω type suffer from a problematic wall boundary condition ($\omega \to \infty$ as $y \to 0$).

8.2.4 Behaviour of Linear Eddy-Viscosity Models in Simple Shear

In simple shear flow the shear stress is

$$-\rho\overline{uv} = \mu_t \frac{\partial U}{\partial y}$$



The three normal stresses are predicted to be equal:

$$\overline{u^2} = \overline{v^2} = \overline{w^2} = \frac{2}{3}k$$

whereas, in practice, there is considerable *anisotropy*; e.g. in the log-law region:

$$\overline{u^2} : \overline{v^2} : \overline{w^2} \approx 1.0 : 0.4 : 0.6$$

Actually, *in simple shear* flows, this is not a problem, since only the gradient of the shear stress $\rho\overline{uv}$ plays a dynamically-significant role in the mean-momentum equation. However, it is a warning of more serious problems in *complex* flows.

8.3 Non-Linear Eddy-Viscosity Models

8.3.1 General Form

The stress-strain relationship for linear eddy-viscosity models gives for the *deviatoric* Reynolds stress (i.e. subtracting the trace):

$$\overline{u_i u_j} - \frac{2}{3}k\delta_{ij} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

Dividing by k and writing $\nu_t = C_\mu k^2/\varepsilon$ gives

$$\frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} = -C_\mu \frac{k}{\varepsilon} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \qquad (10)$$

We define the LHS of (10) as the *anisotropy tensor a_{ij}*; it is the dimensionless and traceless form of the Reynolds stress:

$$a_{ij} \equiv \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} \qquad (11)$$

For the RHS of (10), the symmetric and antisymmetric parts of the mean-velocity gradient are called the *mean strain* and *mean vorticity* tensors, respectively:

$$S_{ij} \equiv \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}), \qquad \Omega_{ij} \equiv \frac{1}{2}(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}) \qquad (12)$$

These can be made non-dimensional using the turbulent timescale k/ε. Using lower case for the non-dimensional forms:

$$s_{ij} \equiv \frac{k}{\varepsilon} S_{ij}, \qquad \omega_{ij} \equiv \frac{k}{\varepsilon} \Omega_{ij} \qquad (13)$$

Equation (10) can then be written in the simpler form

$$a_{ij} = -2C_\mu s_{ij}$$

or,

$$\mathbf{a} = -2C_\mu \mathbf{s} \qquad (14)$$

Hence, the constitutive relation for linear eddy-viscosity models simply says:

"*anisotropy tensor is proportional to dimensionless mean strain*"

The main idea of non-linear eddy-viscosity models is to generalise this to a *non-linear* relationship between the anisotropy tensor and the mean strain and vorticity:

$$\mathbf{a} = -2C_\mu \mathbf{s} + \mathbf{NL}(\mathbf{s}, \boldsymbol{\omega}) \qquad (15)$$

Additional non-linear components cannot be completely arbitrary, but must be symmetric and traceless. For example a quadratic NLEVM must be of the form

$$\begin{aligned} \mathbf{a} &= -2C_\mu \mathbf{s} \\ &\quad + \beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}) \end{aligned} \qquad (16)$$

where $\{.\}$ denotes a trace and $\mathbf{I}$ is the identity matrix:

$$\{\mathbf{M}\} \equiv trace(\mathbf{M}) \equiv M_{ii}, \qquad (\mathbf{I})_{ij} \equiv \delta_{ij} \qquad (17)$$

We shall see below that an appropriate choice of the coefficients β_1, β_2 and β_3 allows the model to reproduce the correct anisotropy in simple shear.

Theory (based on the Cayley-Hamilton Theorem) shows that the most general relationship involves ten independent tensor bases and includes terms up to the 5th power in $\mathbf{s}$ and $\boldsymbol{\omega}$:

$$\mathbf{a} = \sum_{\alpha=1}^{10} C_\alpha \mathbf{T}_\alpha(\mathbf{s}, \boldsymbol{\omega}) \tag{18}$$

where all $\mathbf{T}_\alpha$ are linearly-independent, symmetric, traceless, second-rank tensor products of $\mathbf{s}$ and $\boldsymbol{\omega}$. One possible choice of bases (but by no means the only one) is

Linear:	$\mathbf{T}_1 = \mathbf{s}$
Quadratic:	$\mathbf{T}_2 = \mathbf{s}^2 - \frac{1}{3}\{\mathbf{s}^2\}\mathbf{I}$
	$\mathbf{T}_3 = \boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}$
	$\mathbf{T}_4 = \boldsymbol{\omega}^2 - \frac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}$
Cubic:	$\mathbf{T}_5 = \boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}\mathbf{s} - \frac{2}{3}\{\boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}\}\mathbf{I}$
	$\mathbf{T}_6 = \boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega}$
Quartic:	$\mathbf{T}_7 = \boldsymbol{\omega}^2\mathbf{s}^2 + \mathbf{s}^2\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}(\mathbf{s}^2 - \frac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) - \frac{2}{3}\{\mathbf{s}^2\boldsymbol{\omega}^2\}\mathbf{I}$
	$\mathbf{T}_8 = \mathbf{s}^2\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}\mathbf{s}^2 - \frac{1}{2}\{\mathbf{s}^2\}(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega})$
	$\mathbf{T}_9 = \boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}\boldsymbol{\omega} - \frac{1}{2}\{\boldsymbol{\omega}^2\}(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega})$
Quintic:	$\mathbf{T}_{10} = \boldsymbol{\omega}\mathbf{s}^2\boldsymbol{\omega}^2 - \boldsymbol{\omega}^2\mathbf{s}^2\boldsymbol{\omega}$

Exercise. (i) Prove that all these bases are symmetric and traceless.
(ii) Show that bases $\mathbf{T}_5$ – $\mathbf{T}_{10}$ vanish in 2-d incompressible flow.

The first base corresponds to a linear eddy-viscosity model and the next three to the quadratic extension in equation (16). $\mathbf{T}_5$, $\mathbf{T}_7$, $\mathbf{T}_8$, $\mathbf{T}_9$ contain multiples of earlier bases and hence could be replaced by simpler forms; however, the bases chosen here ensure that they vanish in 2-d incompressible flow.

A number of routes have been taken in devising such NLEVMs, including:

- assuming the form of the series expansion to quadratic or cubic order and simply calibrating against important flows (e.g. Speziale, 1987; Craft, Launder and Suga, 1996);
- simplifying a differential stress model by an explicit solution (e.g. Speziale and Gatski, 1993) or by successive approximation (e.g. Apsley and Leschziner, 1998);
- renormalisation group methods (e.g. Rubinstein and Barton, 1990);
- direct interaction approximation (e.g Yoshizawa, 1987).

In devising such NLEVMs, model developers have sought to incorporate such physically-significant properties as *realisability*:

$$\begin{aligned} &\overline{u_\alpha^2} \geq 0 \qquad &&\text{(positive normal stresses)} \\ &\frac{\overline{u_\alpha u_\beta}^2}{\overline{u_\alpha^2}\;\overline{u_\beta^2}} \leq 1 \qquad &&\text{(Cauchy – Schwartz inequality)} \end{aligned} \tag{19}$$

8.3.2 Cubic Eddy-Viscosity Models

The preferred level of modelling at the University of Manchester is a *cubic* eddy viscosity model, which can be written in the form

$$\begin{aligned}\mathbf{a} = \;& -2C_\mu f_\mu \mathbf{s} \\ & +\beta_1(\mathbf{s}^2 - \tfrac{1}{3}\{\mathbf{s}^2\}\mathbf{I}) + \beta_2(\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega}) + \beta_3(\boldsymbol{\omega}^2 - \tfrac{1}{3}\{\boldsymbol{\omega}^2\}\mathbf{I}) \\ & -\gamma_1\{\mathbf{s}^2\}\mathbf{s} - \gamma_2\{\boldsymbol{\omega}^2\}\mathbf{s} - \gamma_3(\boldsymbol{\omega}^2\mathbf{s} + \mathbf{s}\boldsymbol{\omega}^2 - \{\boldsymbol{\omega}^2\}\mathbf{s} - \tfrac{2}{3}\{\boldsymbol{\omega}\mathbf{s}\boldsymbol{\omega}\}\mathbf{I}) - \gamma_4(\boldsymbol{\omega}\mathbf{s}^2 - \mathbf{s}^2\boldsymbol{\omega})\end{aligned} \tag{20}$$

Note the following properties (some of which will be developed further below and on the example sheet).

(i) A *cubic* stress-strain relationship is the minimum order with at least the same number of independent coefficients as the anisotropy tensor (i.e. 5). In this case it will be precisely 5 if we assume $\beta_3 = 0$ (see (vi) below) and note that the γ_1 and γ_2 terms are tensorially similar to the linear term (see (iv) below).

(ii) The first term on the RHS corresponds to a linear eddy-viscosity model.

(iii) The various non-linear terms evoke sensitivities to specific types of strain:
   – the quadratic (β_1, β_2, β_3) terms evoke sensitivity to *anisotropy*;
   – the cubic γ_1 and γ_2 terms evoke sensitivity to *curvature*;
   – the cubic γ_4 term evokes sensitivity to *swirl*.

(iv) The γ_1 and γ_2 terms are tensorially proportional to the linear term; however they (or rather their difference) provide a sensitivity to curvature, so have been kept distinct.

(v) The γ_3 and γ_4 terms vanish in 2-d incompressible flow.

(vi) Theory and experiment indicate that pure rotation generates no turbulence. This implies that β_3 ought to be 0, at least in the limit $\bar{S} \to 0$.

As an example of such a model we cite the Craft et al. (1996) model in which coefficients are functions of the mean-strain invariants and turbulent Reynolds number:

$$C_\mu = \frac{0.3[1 - \exp(-0.36 e^{0.75\eta})]}{1 + 0.35\eta^{3/2}}$$

$$f_\mu = 1 - \exp[-(\frac{R_t}{90})^{1/2} - (\frac{R_t}{400})^2], \qquad R_t = \frac{k^2}{\nu\tilde{\varepsilon}} \tag{21}$$

where

$$\bar{S} = \sqrt{2S_{ij}S_{ij}}, \quad \bar{\Omega} = \sqrt{2\Omega_{ij}\Omega_{ij}}, \quad \eta = \frac{k}{\tilde{\varepsilon}}\max(\bar{S}, \bar{\Omega}) \tag{22}$$

The coefficients of the non-linear terms are (in the present notation):

$$(\beta_1, \beta_2, \beta_3) = (-0.4,\ 0.4,\ -1.04)C_\mu f_\mu$$

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (40,\ 40,\ 0,\ -80)C_\mu^3 f_\mu \tag{23}$$

Non-linearity is built into both tensor products and strain-dependent coefficients – notably C_μ. The model is completed by transport equations for k and $\tilde{\varepsilon}$. Mean strain and vorticity are non-dimensionalised using $\tilde{\varepsilon}$ rather than ε.

8.3.3 General Properties of Non-Linear Eddy-Viscosity Models

(i) 2-d Incompressible Flow

The non-linear combinations of $\mathbf{s}$ and $\boldsymbol{\omega}$ have particularly simple forms in 2-d incompressible flow. In such a flow:

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Incompressibility ($s_{11} = -s_{22}$) and the symmetry and antisymmetry properties of s_{ij} and ω_{ij} ($s_{21} = s_{12}$, $\omega_{21} = -\omega_{12}$) reduce these to

$$\mathbf{s} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & -s_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 & \omega_{12} & 0 \\ -\omega_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From these we find

$$\mathbf{s}^2 = (s_{11}^2 + s_{12}^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega}^2 = -\omega_{12}^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\boldsymbol{\omega}\mathbf{s} - \mathbf{s}\boldsymbol{\omega} = 2\omega_{12}s_{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 2\omega_{12}s_{11}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (24)$$

PROPERTY 1
In 2-d incompressible flow:

$$\begin{aligned} \mathbf{s}^2 &= (s_{11}^2 + s_{12}^2)\mathbf{I}_2 &&= \tfrac{1}{2}\{\mathbf{s}^2\}\mathbf{I}_2 \\ \boldsymbol{\omega}^2 &= -\omega_{12}^2\mathbf{I}_2 &&= \tfrac{1}{2}\{\boldsymbol{\omega}^2\}\mathbf{I}_2 \end{aligned} \qquad (25)$$

where $\mathbf{I}_2$ = diag(1,1,0). In particular, taking tensor products of $\mathbf{s}^2$ or $\boldsymbol{\omega}^2$ with matrices whose third row and third column are all zero has the same effect as multiplication by the scalars $\tfrac{1}{2}\{\mathbf{s}^2\}$ or $\tfrac{1}{2}\{\boldsymbol{\omega}^2\}$ respectively.

PROPERTY 2

$$\frac{P^{(k)}}{\varepsilon} = -a_{ij}s_{ij} = -\{\mathbf{as}\} \qquad (26)$$

Moreover, in 2-d incompressible flow the quadratic terms do not contribute to the production of turbulent kinetic energy.

Proof.

$$P^{(k)} = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \quad = -k(a_{ij} + \tfrac{2}{3}\delta_{ij})(S_{ij} + \Omega_{ij})$$

Now $(a_{ij}+\frac{2}{3}\delta_{ij})\Omega_{ij}=0$ since Ω_{ij} is antisymmetric, whilst incompressibility implies $\delta_{ij}S_{ij}=S_{ii}=0$. Hence,

$$P^{(k)}=-ka_{ij}S_{ij}$$

or

$$\frac{P^{(k)}}{\varepsilon}=-a_{ij}s_{ij}=-\{\mathbf{as}\}$$

This is true for any incompressible flow, but, in the 2-d case, multiplying (20) by $\mathbf{s}$, taking the trace and using the results (25) it is found that the contribution of the quadratic terms to $\{\mathbf{as}\}$ is 0.

PROPERTY 3

In 2-d incompressible flow the γ_3- and γ_4-related terms of the non-linear expansion (20) vanish.

Proof. Substitute the results (25) for $\mathbf{s}^2$ and $\boldsymbol{\omega}^2$ into (20).

(ii) Particular Types of Strain

The non-linear constitutive relationship (20) allows the model to mimic the response of turbulence to particular important types of strain.

PROPERTY 4

The quadratic terms yield turbulence *anisotropy* in simple shear:

$$\frac{\overline{u^2}}{k}=\frac{2}{3}+(\beta_1+6\beta_2-\beta_3)\frac{\sigma^2}{12}$$

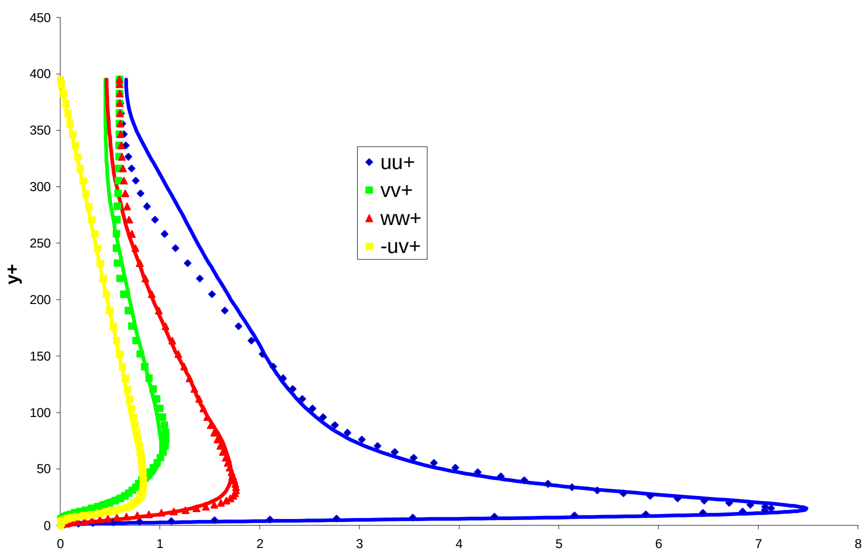
$$\frac{\overline{v^2}}{k}=\frac{2}{3}+(\beta_1-6\beta_2-\beta_3)\frac{\sigma^2}{12} \qquad \text{where} \qquad \sigma=\frac{k}{\varepsilon}\frac{\partial U}{\partial y} \qquad (27)$$

$$\frac{\overline{w^2}}{k}=\frac{2}{3}-(\beta_1-\beta_3)\frac{\sigma^2}{6}$$

This may be deduced by substituting the results (24) into (20), noting that $s_{11}=0$, whilst

$$s_{12}=\omega_{12}=\frac{1}{2}\frac{k}{\varepsilon}\frac{\partial U}{\partial y}=\frac{1}{2}\sigma$$

As an example the figure right shows application of the Apsley and Leschziner (1998) model to computing the Reynolds stresses in channel flow.



PROPERTY 5
The γ_1 and γ_2-related cubic terms yield the correct sensitivity to *curvature*.

In curved shear flow, $\dfrac{\partial U}{\partial y}=\dfrac{\partial U_s}{\partial R}$, $\dfrac{\partial V}{\partial x}=-\dfrac{U_s}{R_c}$, where R_c is radius of curvature. From (24),

$$\{\mathbf{s}^2\}+\{\boldsymbol{\omega}^2\}\equiv 2(s_{12}^2-\omega_{12}^2)$$

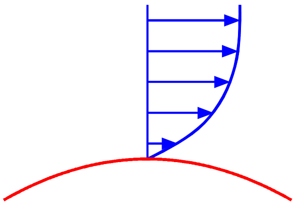
where

$$s_{12}=\frac{1}{2}\left(\frac{\partial U_s}{\partial R}-\frac{U_s}{R_c}\right), \qquad \omega_{12}=\frac{1}{2}\left(\frac{\partial U_s}{\partial R}+\frac{U_s}{R_c}\right)$$

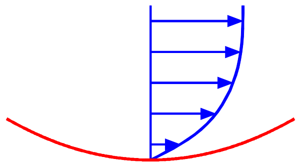
Hence,

$$\{\mathbf{s}^2\}+\{\boldsymbol{\omega}^2\}\equiv -2(\frac{k}{\varepsilon})^2\frac{\partial U_s}{\partial R}\frac{U_s}{R_c}$$

Inspection of the production terms in the stress-transport equations (Section 7.4) shows that curvature is stabilising (reducing turbulence) if U_s increases in the direction away from the centre of curvature ($\partial U_s/\partial R > 0$) and destabilising (increasing turbulence) if U_s decreases in the direction away from the centre of curvature ($\partial U_s/\partial R < 0$). In the constitutive relation (20) the response is correct if γ_1 and γ_2 are both positive.

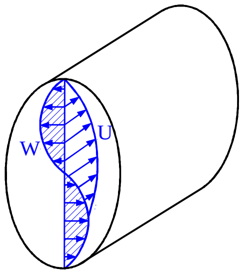


'stable' curvature
(reducing turbulence)



'unstable' curvature
(increasing turbulence)

PROPERTY 6
In 3-d flows, the γ_4-related term evokes the correct sensitivity to *swirl*.



8.4 Differential Stress Modelling

Differential stress models (aka *Reynolds-stress transport models* or *second-order closure*) solve a separate scalar-transport equation for each stress component $\overline{u_i u_j}$:

$$\rho \frac{\mathrm{D}(\overline{u_i u_j})}{\mathrm{D}t} = \frac{\partial d_{ijk}}{\partial x_k} + \rho(P_{ij} + F_{ij} + \Phi_{ij} - \varepsilon_{ij}) \tag{28}$$

(For a derivation see the course notes for the "Boundary Layers" module).

Such models, in principle, contain much more turbulence physics because the rate-of-change, advection and production terms are exact. The nearest thing to a standard model is a high-Re closure based on that of Launder et al. (1975) and Gibson and Launder (1978).

Term	Name and role	Model
$\rho \frac{\mathrm{D}(\overline{u_i u_j})}{\mathrm{D}t}$	**RATE OF CHANGE (time derivative + advection)** Transport with the mean flow.	EXACT
P_{ij}	**PRODUCTION (mean strain)** Generation of turbulence energy from the mean flow.	EXACT $P_{ij} \equiv -\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k}$
F_{ij}	**PRODUCTION (body forces)** Generation of turbulence energy by body forces.	EXACT (in principle) $F_{ij} \equiv \overline{f_i u_j} + \overline{f_j u_i}$
d_{ijk}	**DIFFUSION** Spatial redistribution.	$d_{ijk} = (\mu\delta_{kl} + C_s \frac{\rho k \overline{u_k u_l}}{\varepsilon}) \frac{\partial}{\partial x_l}(\overline{u_i u_j})$
Φ_{ij}	**PRESSURE-STRAIN** Redistribution of turbulence energy between components.	$\Phi_{ij} = \Phi_{ij}^{(1)} + \Phi_{ij}^{(2)} + \Phi_{ij}^{(w)}$ $\Phi_{ij}^{(1)} = -C_1 \frac{\varepsilon}{k}(\overline{u_i u_j} - \tfrac{2}{3} k \delta_{ij})$ $\Phi_{ij}^{(2)} = -C_2 (P_{ij} - \tfrac{1}{3} P_{kk} \delta_{ij})$ $\Phi_{ij}^{(w)} = (\tilde{\Phi}_{kl} n_k n_l \delta_{ij} - \tfrac{3}{2} \tilde{\Phi}_{ik} n_j n_k - \tfrac{3}{2} \tilde{\Phi}_{jk} n_i n_k) f$ $\tilde{\Phi}_{ij} = C_1^{(w)} \frac{\varepsilon}{k} \overline{u_i u_j} + C_2^{(w)} \Phi_{ij}^{(2)}, \quad f = \frac{C_\mu^{3/4} k^{3/2}/\varepsilon}{\kappa y_n}$
ε_{ij}	**DISSIPATION** Removal of turbulence energy by viscosity	$\varepsilon_{ij} = \tfrac{2}{3} \varepsilon \delta_{ij}$

Typical values of the constants are:

$$C_1 = 1.8\,, \qquad C_2 = 0.6\,, \qquad C_1^{(w)} = 0.5\,, \qquad C_2^{(w)} = 0.3 \tag{29}$$

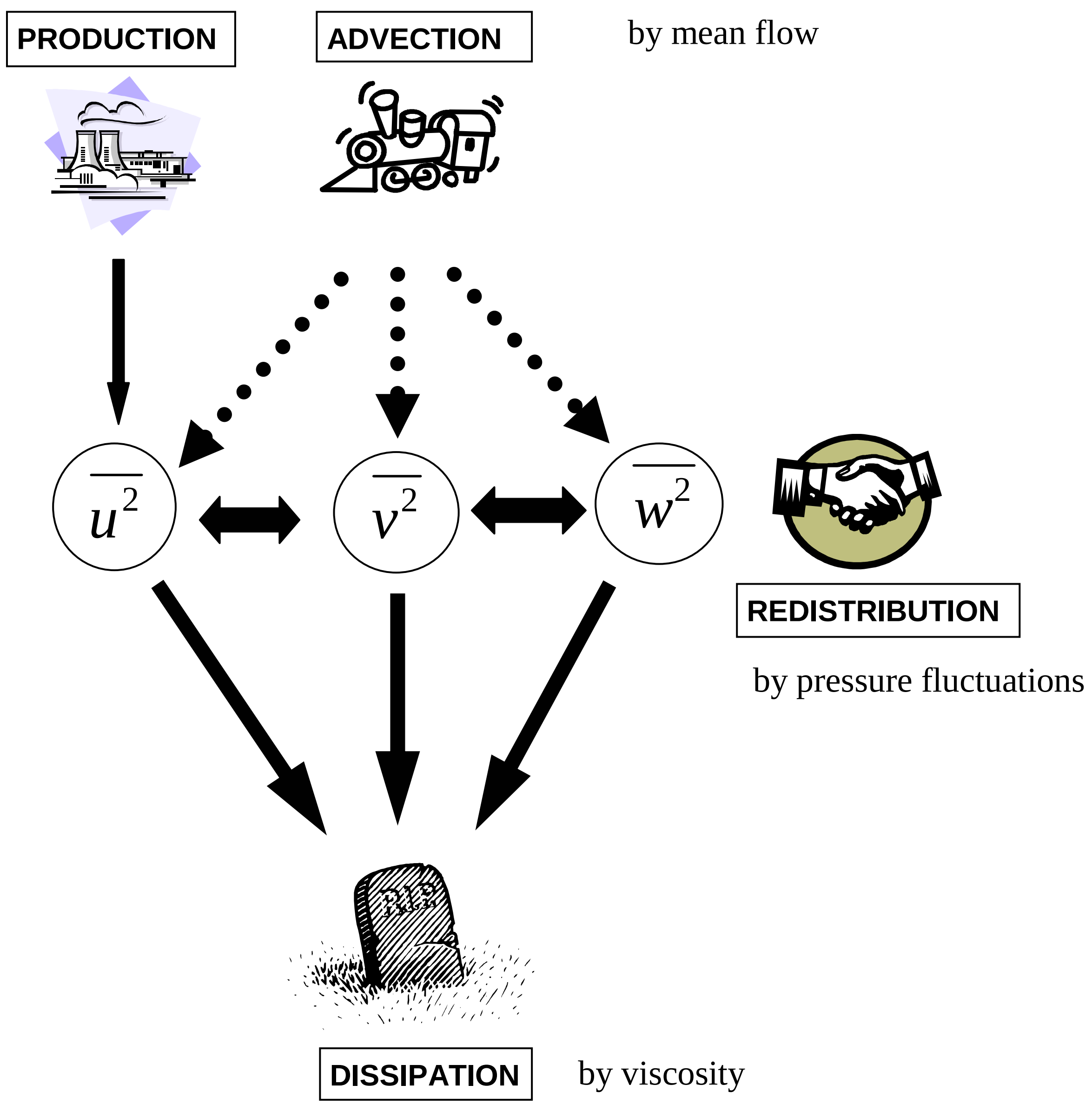
Energy in Turbulent Fluctuations

In simple shear flow (where $\partial U/\partial y$ is the only non-zero mean-velocity gradient) the production terms of the normal stresses are:

$$P_{11} = -2uv\frac{\partial U}{\partial y}, \qquad P_{22} = P_{33} = 0$$

Hence, **production** of turbulence energy predominantly feeds the $\overline{u^2}$ component. Energy is then transferred to fluctuations in the cross-stream directions by the **redistributive** effect of pressure fluctuations. At small scales local gradients are sufficiently large for viscosity to **dissipate** turbulent energy.

There is a continual **energy cascade** from the energy entering the turbulence at the large scales of the flow, though shear instabilities continually producing eddies at smaller scales, until ultimately energy is removed by viscosity.



The stress-transport equations must be supplemented by a means of specifying ε – typically by its own transport equation, or one for a related quantity such as ω.

As is suggested by the table, the most significant term requiring modelling is the pressure-strain correlation (which is formed, in practice, by the average product of pressure fluctuations and fluctuating velocity gradients). This term is traceless (i.e. the sum of the diagonal terms $\Phi_{11} + \Phi_{22} + \Phi_{33} = 0$) and its accepted role is to promote isotropy – hence the form of model for $\Phi_{ij}^{(1)}$ and $\Phi_{ij}^{(2)}$. Near walls this isotropising tendency must be over-ridden, necessitating a "wall-correction" term $\Phi_{ij}^{(w)}$.

Where body forces are present (e.g. in buoyant or rotating flows) additional production terms must be included.

General Assessment of DSMs

For:

- Include more turbulence physics than eddy-viscosity models.
- Advection and production terms ("energy-in" terms) are exact and do not need modelling.

Against:

- Models are very complex and many important terms (particularly the redistribution and dissipation terms) require modelling.
- Models are very expensive computationally (6 stress-transport equations in 3 dimensions) and tend to be numerically unstable (only the small molecular viscosity contributes to any sort of gradient diffusion term).

Other DSMs of Interest

- Speziale et al. (1991) – non-linear Φ_{ij} formulation, eliminating wall-correction terms;
- Craft (1998) – low-Re DSM, attempting to eliminate wall-dependent parameters;
- Jakirlić and Hanjalić (1995) – low-Re DSM admitting anisotropic dissipation;
- Wilcox (1988b) – low-Re DSM, with ω rather than ε as additional turbulent scalar.

Excellent references for developments in Reynolds-stress transport modelling can be found in Launder (1989) and Hanjalić (1994).

8.5 Implementation of Turbulence Models in CFD

8.5.1 Transport Equations

The implementation of a turbulence model in CFD requires:

(1) a means of specifying the turbulent stresses $\overline{u_i u_j}$, by either:
- a constitutive relation (eddy-viscosity models), or
- individual transport equations (differential stress models);

(2) the solution of additional scalar-transport equations.

Special Considerations for the Mean Flow Equations

- $\rho\overline{u_i u_j}$ represents a turbulent flux of U_i-momentum in the x_j direction, but only a part of this can be treated implicitly as a diffusion-like term. e.g. for the U equation through a face normal to the y direction:

$$-\rho\overline{uv} = \mu_t(\underbrace{\frac{\partial U}{\partial y}}_{\substack{diffusive\\part}} + \underbrace{\frac{\partial V}{\partial x}) + (non-linear\ terms)}_{transferred\ to\ source}$$

  The non-diffusive part of the flux is transferred to the source term (and treated explicitly – i.e. held constant for that iteration). Nevertheless, it is still treated in a conservative fashion; i.e. it is worked out on a cell face so that the mean momentum lost by one cell is equal to that gained by its neighbour.

- The lack of a turbulent viscosity in differential stress models can lead to numerical instability. This can be addressed by the use of "effective viscosities" – see below.

Special Considerations for the Turbulence Equations

- They are usually source-dominated; i.e. the most significant terms are production, redistribution and dissipation; (this is sometimes used as an excuse for a low-order advection scheme).

- Variables such as k and ε must be non-negative. This demands:
  - care in discretising the source term (see below);
  - use of an unconditionally-bounded advection scheme.

Source-Term Linearisation For Non-Negative Quantities

The general discretised scalar-transport equation for a control volume centred on node P is

$$a_P\phi_P - \sum_F a_F\phi_F = b_P + s_P\phi_P$$

For stability one requires

$$s_P \le 0$$

To ensure non-negative ϕ one requires, in addition,

$$b_P \geq 0$$

You should, by inspection of the k and ε transport equations (3), be able to identify how the source term is linearised in this way, with one positive part and one negative part, the latter preferably proportional to the transported variable, k or ε.

If $b_P < 0$ for a quantity such as k or ε which is always non-negative (e.g. due to transfer of non-linear parts of the advection term or non-diffusive fluxes to the source term) then, to ensure that the variable doesn't become negative, the source term should be rearranged as

$$s_P \to s_P + (\frac{b_P}{\phi_P^*})\phi_P \qquad (30)$$

$$b_P \to 0$$

where * denotes the current value of a variable.

8.5.2 Wall Boundary Conditions

At walls the no-slip boundary condition applies, so that both mean and fluctuating velocities vanish. At high Reynolds numbers this presents three problems:

- there are very large flow gradients;
- wall-normal fluctuations are suppressed (i.e. selectively damped);
- viscous and turbulent stresses are of comparable magnitude.

There are two main ways of handling this in turbulent flow:

- **low-Reynolds-number turbulence models**
  – resolve the flow right up to the wall with a very fine grid and viscous modifications to the turbulence equations to ensure the correct near-wall rather than log-layer behaviour;
- **wall functions**
  – use a coarser grid and assume theoretical profiles between the near-wall node and the boundary.

<u>**Low-Reynolds-Number Turbulence Models**</u>

- Aim to resolve the flow right up to the boundary.

- Have to include effects of molecular viscosity in the coefficients of the eddy-viscosity formula and ε (or ω) transport equations.

- Try to ensure the theoretical near-wall behaviour:

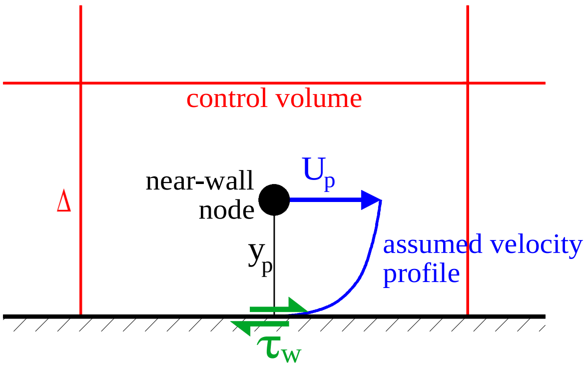
$$k \propto y^2, \qquad \varepsilon \sim \frac{2\nu k}{y^2} \sim \text{constant}, \qquad \nu_t \propto y^3 \qquad (y \to 0) \qquad (31)$$

- Full resolution of the flow requires the near-wall node to satisfy $y^+ \leq 1$, where

$$y^+ \equiv \frac{u_\tau y}{\nu}, \qquad u_\tau = \sqrt{\tau_w/\rho} \qquad (32)$$

  This can be very computationally demanding, particularly for high-speed flows.

High-Reynolds-Number Turbulence Models



- Bridge the near-wall region with *wall functions*; i.e. assume profiles (based on boundary-layer theory) between near-wall node and boundary.

- OK if near equilibrium (e.g. slowly-developing boundary layers), but dodgy in highly non-equilibrium regions (particularly near impingement, separation or reattachment points).

- The near-wall node should ideally be placed in the region $30 < y^+ < 50$ (range 15 -150 generally acceptable). This means that numerical meshes cannot be arbitrarily refined close to solid boundaries.

In the finite-volume method, various quantities are required from the wall-function approach. Values may be fixed on the wall (*w*) itself or by forcing a value at the near-wall node (*P*).

Variable	**Wall boundary condition**	**Required from wall function**
Mean velocity (U,V,W)	(relative) velocity = 0 at the wall	Wall shear stress
k, $\overline{u_i u_j}$	$\overline{u_i u_j} = k = 0$ at the wall; zero flux	Cell-averaged production and dissipation
ε	ε_P fixed at near-wall node	Value at the near-wall node

The means of deriving these quantities are set out below.

Mean-Velocity Equation: Wall Shear Stress

The friction velocity u_τ is defined in terms of the wall shear stress:

$$\tau_w = \rho u_\tau^2$$

If the near-wall node lies in the logarithmic region then

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln(E y_P^+) , \qquad y_P^+ = \frac{y_P u_\tau}{\nu} \tag{33}$$

where subscript *P* denotes the near-wall node. Given the value of U_P this could be solved (iteratively) for u_τ and hence the wall stress τ_w.

However, a better approach when the turbulence is clearly far from equilibrium (e.g. near separation or reattachment points) is to estimate an "equivalent" friction velocity *from the turbulent kinetic energy*:

$$u_0 = C_\mu^{1/4} k_P^{1/2}$$

and integrate the mean-velocity profile assuming an eddy viscosity ν_t. If we adopt the log-law version:

$$\nu_t = \kappa u_0 y$$

and solve for U from

$$\tau_w = \rho \nu_t \frac{\partial U}{\partial y}$$

we get

$$\tau_w / \rho = \frac{\kappa u_0 U_P}{\ln(E \dfrac{y_P u_0}{\nu})} \tag{34}$$

(If the turbulence were genuinely in equilibrium, then u_0 would equal u_τ and (33) and (34) would be equivalent).

A better approach is to assume a total viscosity (molecular + eddy) which matches both the viscous ($\nu_{eff} = \nu$) and log-layer ($\nu_{eff} \sim \kappa u_\tau y$) limits:

$$\nu_{eff} = \nu + \max\{0, \kappa u_0 (y - y_\nu)\} \tag{35}$$

where y_ν is a matching height. Similar integration to before leads to both viscous sublayer and log-law limits

$$\frac{U}{u_0} = \frac{\tau_w}{\rho u_0^2} \times \begin{cases} y^+, & y^+ \le y_\nu^+ \\ y_\nu^+ + \dfrac{1}{\kappa} \ln\{1 + \kappa(y^+ - y_\nu^+)\}, & y^+ \ge y_\nu^+ \end{cases}, \qquad y^+ \equiv \frac{y u_0}{\nu} \tag{36}$$

where we note that y^+ is based on u_0 rather than the unknown u_τ. A similar approach can be applied for rough-wall boundary layers (Apsley, 2007), where y_ν^+ is a function of roughness. A typical (smooth-wall) value of y_ν^+ is 7.37.

As far as the computational implementation is concerned the required output for a finite-volume calculation is the wall shear stress in terms of the mean velocity at the near-wall node, y_p, not vice versa. To this end, (36) is conveniently rearranged in terms of an *effective wall viscosity* $\nu_{eff,wall}$ such that

$$\tau_w = \rho \nu_{eff,wall} \frac{U_p}{y_p} \tag{37}$$

where

$$\nu_{eff,wall} = \nu \times \begin{cases} 1, & y_P^+ \le y_\nu^+ \\ \dfrac{y_P^+}{y_\nu^+ + \dfrac{1}{\kappa} \ln\{1 + \kappa(y_P^+ - y_\nu^+)\}}, & y_P^+ \ge y_\nu^+ \end{cases} \tag{38}$$

k **Equation: Cell-Averaged Production and Dissipation**

The source term of the k transport equation requires *cell-averaged* values of production $P^{(k)}$ and dissipation rate ε. These are derived by assuming profiles for these quantities:

$$P^{(k)} \equiv -\overline{uv} \frac{\partial U}{\partial y} = \begin{cases} 0 & y \le y_\nu \\ \nu_t (\dfrac{\partial U}{\partial y})^2 & y > y_\nu \end{cases} \qquad \text{where } \nu_t = \nu_{eff} - \nu \tag{39}$$

$$\varepsilon = \begin{cases} \varepsilon_w & (y \le y_\varepsilon) \\ \dfrac{u_0^3}{\kappa(y - y_d)} & (y > y_\varepsilon) \end{cases} \tag{40}$$

where, for *smooth* walls, the matching height y_ε and offset y_d are given in wall units by (see Apsley, 2007):

$$y_\varepsilon^+ = 27.4\,, \qquad y_d^+ = 4.9$$

Integration over a cell (see example sheet) then leads to cell averages

$$P_{av}^{(k)} \equiv \frac{1}{\Delta}\int_0^\Delta P^{(k)} \;\mathrm{d}y \;=\frac{(\tau_w/\rho)^2}{\kappa u_0 \Delta}\left\{\ln[1+\kappa(\Delta^+ - y_\nu^+)] - \frac{\kappa(\Delta^+ - y_\nu^+)}{1+\kappa(\Delta^+ - y_\nu^+)}\right\} \qquad (41)$$

$$\varepsilon_{av} = \frac{1}{\Delta}\int_0^\Delta \varepsilon \;\mathrm{d}y \;=\frac{u_0^3}{\kappa\Delta}\left[\ln(\frac{\Delta}{y_\varepsilon})+1\right] \qquad (42)$$

ε Equation: Boundary Condition on ε

ε_P is fixed from its assumed profile (equation (40)) at the near-wall node. A particular value at a cell centre can be forced in a finite-volume calculation by modifying the source coefficients:

$$s_P \to -\gamma, \qquad b_P \to \gamma\varepsilon_P$$

where γ is a large number (e.g. 10^{30}). The matrix equations for that cell then become

$$(\gamma + a_P)\phi_P - \sum a_F \phi_F = \gamma\varepsilon_P$$

or

$$\phi_P = \frac{\sum a_F \phi_F}{\gamma + a_P} + \frac{\gamma}{\gamma + a_P}\varepsilon_P$$

Since γ is a large number this effectively forces ϕ_P to take the value ε_P.

Reynolds-Stress Equations

For the Reynolds stresses, one method is to fix the values at the near-wall node from the near-wall value of k and the *structure functions* $\overline{u_i u_j}/k$, the latter being derived from the differential stress-transport equations on the assumption of local equilibrium. For the standard model this gives (see the example sheet):

$$\begin{aligned}
\frac{\overline{v^2}}{k} &= \frac{2}{3}\left(\frac{-1+C_1+C_2-2C_2^{(w)}C_2}{C_1+2C_1^{(w)}}\right) \\
\frac{\overline{u^2}}{k} &= \frac{2}{3}\left(\frac{2+C_1-2C_2+C_2^{(w)}C_2}{C_1}\right) + \frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k} \\
\frac{\overline{w^2}}{k} &= \frac{2}{3}\left(\frac{-1+C_1+C_2+C_2^{(w)}C_2}{C_1}\right) + \frac{C_1^{(w)}}{C_1}\frac{\overline{v^2}}{k} \\
-\frac{\overline{uv}}{k} &= \sqrt{\left(\frac{1-C_2+\frac{3}{2}C_2^{(w)}C_2}{C_1+\frac{3}{2}C_1^{(w)}}\right)\frac{\overline{v^2}}{k}}
\end{aligned} \qquad (43)$$

With the values for C_1, C_2, etc. from the standard model this gives

$$\frac{\overline{u^2}}{k}=1.098\,,\qquad \frac{\overline{v^2}}{k}=0.248\,,\qquad \frac{\overline{w^2}}{k}=0.654\,,\qquad \frac{-\overline{uv}}{k}=0.255 \tag{44}$$

When the near-wall flow and wall-normal direction are not conveniently aligned in the *x* and *y* directions respectively, the actual structure functions can be obtained by rotation. However, for 3-dimensional and separating/reattaching flow the *flow-oriented* coordinate system is not fixed *a priori* and can swing round significantly between iterations if the mean velocity is small, making convergence difficult to obtain. A second – and now my preferred – approach (Apsley, 2007) is to use cell-averaged production and dissipation in the Reynolds-stress equations in the same manner as the *k*-equation, noting that, in simple shear and in flow-aligned coordinates:

$$P_{11}=-2\overline{uv}\frac{\partial U}{\partial y}\quad =2P^{(k)}\,,\qquad P_{22}=P_{33}=0$$

$$P_{12}=-\overline{v^2}\frac{\partial U}{\partial y}\quad =\frac{\overline{v^2}}{\overline{uv}}P^{(k)}\,,\qquad P_{23}=P_{31}=0$$

with the ratio $\overline{v^2}/\overline{uv}$ determined from (44) as –0.97. In the wall-function formulation, $P^{(k)}$ is proportional to the square of the velocity at the near-wall node, so rotating from flow-aligned coordinates to the actual Cartesian coordinate system does not cause discontinuities in the stress production where the velocity reverses sign; e.g. near separation or reattachment points.

8.5.3 Effective Viscosity for Differential Stress Models

DSMs contain no turbulent viscosity and have a reputation for numerical instability.

An artificial means of promoting stability is to add and subtract a gradient-diffusion term to the turbulent flux:

$$\overline{u_\alpha u_\beta}=(\overline{u_\alpha u_\beta}+\nu_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta})-\nu_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} \tag{45}$$

with the first part averaged between nodal values and the last part discretised across a cell face and treated implicitly; (very similar to the Rhie-Chow algorithm for pressure-velocity coupling in the momentum equations).

The simplest choice for the *effective viscosity* $\nu_{\alpha\beta}$ is just

$$\nu_{\alpha\beta}=\nu_t=C_\mu\frac{k^2}{\varepsilon} \tag{46}$$

A better choice is to make use of a natural linkage between individual stresses and the corresponding mean-velocity gradient which arise from the actual stress-transport equations.

Assuming that the stress-transport equations (with no body forces) are source-dominated then

$$P_{ij}+\Phi_{ij}-\varepsilon_{ij}\approx 0$$

or, with the basic DSM (without wall-reflection terms),

$$P_{ij}-C_1\varepsilon(\frac{\overline{u_iu_j}}{k}-\tfrac{2}{3}\delta_{ij})-C_2(P_{ij}-\tfrac{1}{3}P_{kk}\delta_{ij})-\tfrac{2}{3}\varepsilon\delta_{ij}\approx 0$$

Expand this, identifying the terms which contain only $\overline{u_\alpha u_\beta}$ or $\dfrac{\partial U_\alpha}{\partial x_\beta}$ as follows.

For the normal stresses $\overline{u_\alpha^2}$:

$$P_{\alpha\alpha} - C_1 \frac{\varepsilon}{k}(\overline{u_\alpha^2} - \ldots) - C_2 \tfrac{2}{3} P_{\alpha\alpha} + \ldots = 0$$

Hence,

$$\overline{u_\alpha^2} = \frac{(1-\frac{2}{3}C_2)}{C_1}\frac{k}{\varepsilon}P_{\alpha\alpha} + \ldots \quad = \frac{(1-\frac{2}{3}C_2)}{C_1}\frac{k}{\varepsilon}(-2\overline{u_\alpha^2}\frac{\partial U_\alpha}{\partial x_\alpha} + \ldots)$$

Similarly for the shear stresses $\overline{u_\alpha u_\beta}$:

$$P_{\alpha\beta} - C_1 \frac{\varepsilon}{k}\overline{u_\alpha u_\beta} - C_2 P_{\alpha\beta} + \ldots = 0$$

whence

$$\overline{u_\alpha u_\beta} = \frac{(1-C_2)}{C_1}\frac{k}{\varepsilon}P_{\alpha\beta} + \ldots \quad = \frac{(1-C_2)}{C_1}\frac{k}{\varepsilon}(-\overline{u_\beta^2}\frac{\partial U_\alpha}{\partial x_\beta} + \ldots)$$

Hence, from the stress-transport equations,

$$\begin{aligned} \overline{u_\alpha^2} \quad &= -\nu_{\alpha\alpha}\frac{\partial U_\alpha}{\partial x_\alpha} + \ldots \\ \overline{u_\alpha u_\beta} \quad &= -\nu_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} + \ldots \end{aligned} \qquad (47)$$

where the effective viscosities (both for the U_α component of momentum) are:

$$\nu_{\alpha\alpha} = 2\left(\frac{1-\frac{2}{3}C_2}{C_1}\right)\frac{k\overline{u_\alpha^2}}{\varepsilon}, \qquad \nu_{\alpha\beta} = \left(\frac{1-C_2}{C_1}\right)\frac{k\overline{u_\beta^2}}{\varepsilon} \qquad (48)$$

Note that the effective viscosities are anisotropic, being linked to particular normal stresses.

A more detailed analysis can accommodate wall-reflection terms in the pressure-strain model, but the extra complexity is not justified.

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