The Resilience of the Logarithmic Law to Pressure Gradients: Evidence from DNS

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Wall-bounded turbulence in pressure gradients is studied using a Poiseuille-Couette flow, and Direct Numerical Simulation at appreciable Reynolds numbers. The key motivation is to include adverse pressure gradients, unlike the favorable ones in the well-studied Poiseuille flow. The central question is how the logarithmic law reacts to the gradient in the total shear stress, and ratios of "log region" stress to wall stress, τ^+, ranging from roughly 0.6 to 1.8 have been obtained. The normalized pressure gradients $p^+ \equiv d\tau^+/dy^+$ at the two walls are -0.00056 and $+0.034$. The outcome is in agreement with the experimental findings in boundary layers by Head and Galbraith: the logarithmic velocity profile is vastly more resilient than two other, equally plausible assumptions, namely universality of the mixing length at $l = \kappa y$ and of the eddy viscosity $\nu_t = u_\tau \kappa y$. This is with a unique Karman constant κ, of course. This appears new as a DNS result, and free of the experimental uncertainty over skin-friction. It is probably more remarkable in the adverse gradient, in that the wall shear stress is small and could be expected to be overwhelmed by the stress in the bulk of the flow. We view this finding as somewhat counter-intuitive, but also as quite well-established.

1. Introduction

The cornerstone of our knowledge of wall-bounded turbulent flows, which we would describe as "semi-theoretical," is the logarithmic law. The purest arguments in its favor are that this layer is dominated by the shear stress, equal to u_τ^2 (the density is set to 1, and u_τ is the friction velocity). It is also dominated by the blocking effect of the wall, so that the wall distance y sets the size of the largest, and stress-producing, eddies. An excellent source is Bradshaw & Huang 1995, and they do also discuss the problem at hand here.

This framework applies unquestionably only to a constant-stress layer: u_τ is a wall quantity, but gives the shear stress across the entire layer in which universal behaviour is expected. Thus, $\tau(y) = u_\tau^2$. However, there is keen interest in layers without non-uniform stress, for two reasons. The first is practical, and is the need to predict such flows via turbulence modelling; these are flows with pressure gradients, and of great practical importance. The second is that logarithmic behaviour is observed even in flows with favorable gradients (so that $\tau(y) < u_\tau^2$, often to a ratio of the order of 2/3 or less). Examples include Poiseuille flows in channels and pipes, and the Ekman layer (Spalart et al. 2008). This poses a theoretical challenge of great interest, for the following reason.

The log law can be motivated directly in terms of the velocity profile. If the shear flow is controlled by u_τ and y, dimensional analysis dictates for the shear rate:

$$\frac{dU}{dy} = \frac{u_\tau}{\kappa y} \tag{1}$$

where κ is the Karman constant. Another approach to wall-bounded turbulence, attributed to Clauser, is through the eddy viscosity ν_t. The shear stress is $-\overline{u'v'} = \nu_t dU/dy$. Again by dimensional analysis,

$$\frac{-\overline{u'v'}}{dU/dy} \equiv \nu_t = u_\tau \kappa y. \tag{2}$$

Finally, the mixing length equation is $\nu_t = l^2 dU/dy$, and by dimensional analysis,

$$\frac{\sqrt{-\overline{u'v'}}}{dU/dy} \equiv l = \kappa y. \tag{3}$$

In a constant-stress layer, outside the viscous layer, (1), (2) and (3) are equivalent since $-\overline{u'v'} = \tau(y) = u_\tau^2$.

In spite of the name, equations (2,3) involving eddy viscosity should not necessarily be viewed as "turbulence modeling" in contrast to (1) which would be "theoretical." All three amount to assertions about a length scale being proportional to the wall distance y, with u_τ setting the velocity scale. Note also that (3) has a slightly different nature, in that it does not involve u_τ; it is local in y. This may explain its prominence in algebraic turbulence models, without stating whether the primary reason is physics, or convenience. In any case, none of the three arguments is more rigorous than the other two.

When the shear stress is not constant, equations (1-3) conflict. This has of course been known for a long time, and the desire to determine which argument survives, if any, has been strong. In a series of articles, Galbraith, Sjolander and Head forcefully argued that experimental evidence favors (1) (Galbraith & Head 1975, Galbraith et. al. 1975, Head & Galbraith 1975.) DNS evidence has been in agreement with this, but very limited (Spalart & Watmuff 1993). This was quite important especially as the mixing-length equation (3) has been used so much, but does not seem to have been followed up on, other than by Johnson & Coakley 1990. Naturally, no experimental measurement is perfect, but we have not found discussions which would argue against the conclusions in these three articles based on experimental error.

The circular aspect of these 1975 articles is noted here, as it was then: they use a logarithmic velocity profile to fit the measurements and calculate derivatives, and in the end conclude that the log law is more accurate. However, the fits are used to calculate the shear stress and then the mixing length via the momentum equation, which is not direct and has a chance of "softening" the assumption made. Nevertheless, independent measurement of the skin friction and near-wall profiles would have been much preferable.

It now appears possible to study this question by DNS, without excessively low Reynolds numbers obscuring the findings. While a vast body of Poiseuille-flow results was available, it was felt that cases with adverse pressure gradient (APG) would be essential. Each Couette-Poiseuille flow provides and APG and an FPG case. These flows form a two-parameter family, conveniently described with the Reynolds number Re_τ based on channel half-width h and $u_{\tau,av}$, based on the shear stress at the centerline, and the ratio of shear stress at the two walls, by convention made smaller than 1: the ratio of the stress at the APG wall to that at the FPG wall. In this initial study, we set the stress ratio to 0.3, and Re_τ to 490.

A didactic reason to seek APG cases is the following. Equations (1) and (2) are non-local, which creates the need to explain why the wall value of stress, through u_τ, should dominate in a region where τ is different. In FPG flows, the stress is weaker away from the wall, which could explain why the region closer to the wall dictates the velocity scale. Turbulent kinetic energy and eddy scales from below would be permeating the log layer. In general, Poiseuille-flow DNS has agreed with (1) fairly well, within the setting of limited Reynolds number of course (Hoyas & Jiménez 2006). Ekman-layer DNS has agreed noticeably better (Spalart et al. 2008). Recall that this flow also has a favorable gradient, to the point that logarithmic behaviour is still observed fairly closely when the shear stress has already fallen roughly by 30%. This is enough to bring out the conflict between (1,2,3).

With adverse gradient, the shear stress is weak at the wall, raising the possibility that its influence on the region that is a candidate for universal behaviour would also be weakened. In any case, the principal task here is to produce flows with appreciable deviations of $\tau(y)$ from u_τ^2, in both directions, and to test the three candidate properties (1,2,3). Possible outcomes include a clear "win" for one of the three, or a trend towards an intermediate behaviour. Unclear trends, or conflicting trends between the two walls, could of course not be ruled out. Note that (3) will fall in-between (1) and (2), in terms of deviations from universal behaviour, because

$$\left(\frac{l}{\kappa y}\right) = \sqrt{\left(\frac{u_\tau}{\kappa y dU/dy}\right)\left(\frac{\nu_t}{\kappa y u_\tau}\right)}. \tag{4}$$

In a constant-stress layer, this amounts to $1 = \sqrt{1 \times 1}$. Note also that for linear variations of $\tau(y)$, which are certainly typical of the flows with moderate pressure gradients in which one or another universal behaviour may still be hoped for, the outcome could not be that two of the properties are both satisfied, but with a different value of κ. This is emphatically not a search for "adaptive" κ values. In crude terms, one candidate will succeed, at the most.

2. Numerical Considerations

The simulation is performed using a version of the Fourier/Chebyshev spectral channel code of Kim, Moin & Moser (see Kim et al. 1987), from which it differs algorithmically in the time integration (a Crank-Nicolson/3rd order Runge-Kutta scheme is used in the present work). Moving-wall boundary conditions were also added, with the walls translating at velocities $\pm 20 u_{\tau 0}$, where $u_{\tau 0}$ is a reference friction velocity. The values of Re_τ are based on h, which is also a reference length scale, but not the most relevant, as seen below. Most graphs will use the friction velocity of the relevant wall.

The computational domain is of size $4\pi h \times 2h \times 2\pi h$, in x, y, z respectively, with a grid resolution of $480 \times 241 \times 576$. The resolution is given in wall units in Table 1. These numbers match or better those of Kim et al. 1987, so that accuracy is not a concern (the Runge-Kutta scheme being slightly superior to their Adams-Bashforth scheme). Also, y_{10}^+ is the height of the tenth grid point, Δy_c^+ is the centreline resolution, and Λ^+ the domain size in wall units.

3. Results

Figure 1 shows the Reynolds and total shear stresses across the channel, to exhibit the extent of the variation from one wall to the other, and outline the thickness of the viscous-influenced layers. This will serve as a reference for later figures. The skin-friction ratio of 0.3 is the major design parameter. The left wall will be referred to as "FPG"

TABLE 1. Run parameters in wall units.

Wall	Re_τ	p^+	Λ_x^+	Λ_z^+	Δx^+	Δz^+	y_{10}^+	Δy_c^+	Δt^+
APG	347	0.034	4361	2180	9.1	3.8	2.4	4.5	0.036
FPG	630	−0.00056	7917	3958	16.5	6.9	4.4	8.2	0.12

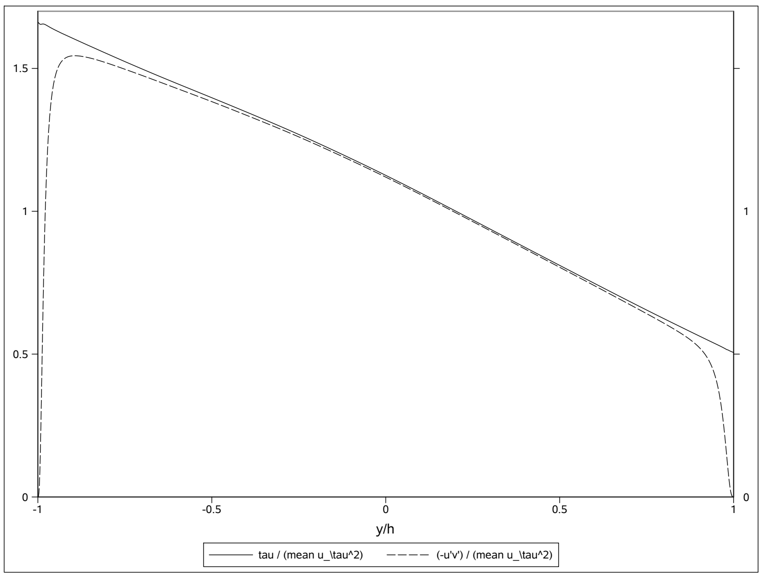

FIGURE 1. Shear stress.

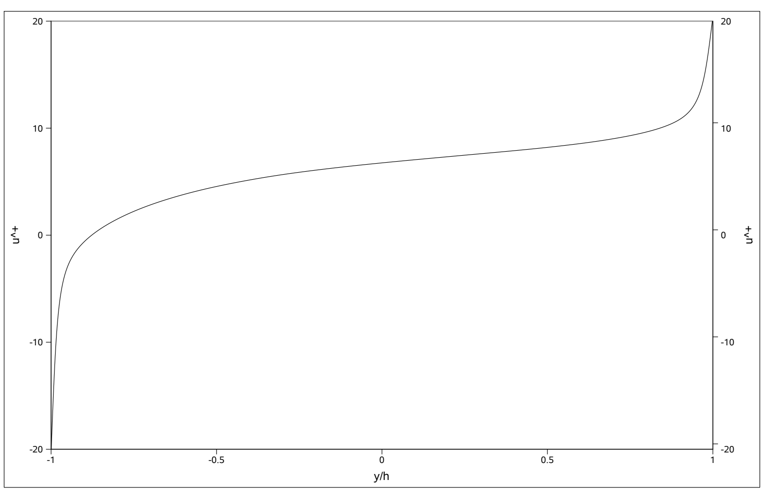

FIGURE 2. Velocity profile.

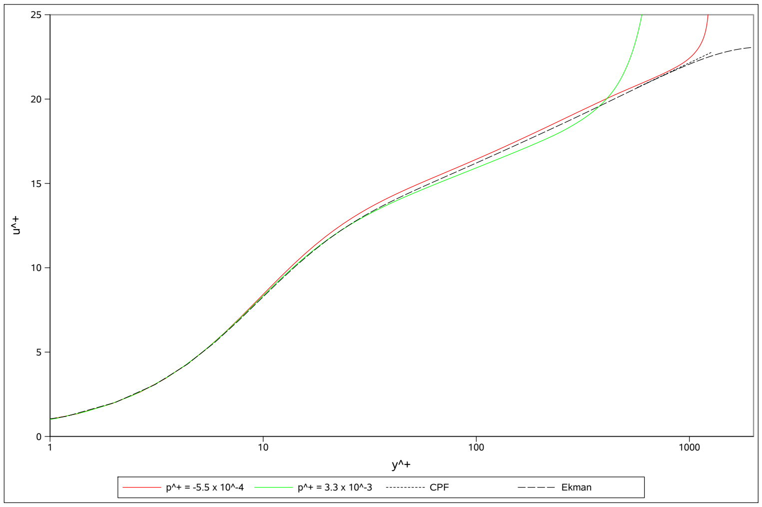

FIGURE 3. Velocity profiles.

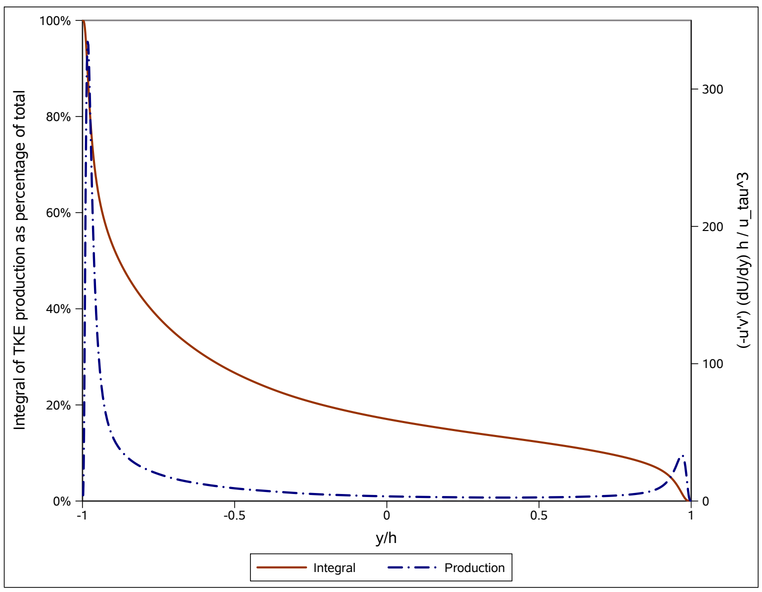

FIGURE 4. Production Integral.

and the right one as "APG." The FPG wall is, at first sight, very similar to a Poiseuille wall, although the shear stress does not reach zero before the influence of the opposite wall is felt. The APG wall layer is more unique. The total stress reaches about twice the wall value before the opposite wall is felt, purely based on intuition. Thus, the pressure gardient is strong, which seemed a sensible choice for this first case. In wall units, the pressure gradient is $+0.034$, compared with -0.00056 for FPG. The Table gives Re_τ values based on channel half-width, but it will be seen that the zone of influence of the FPG wall extends beyond the centerline.

In figure 2, the velocity profile is shown in the same axes, displaying a modified Couette-flow shape, with a shift towards the velocity of the APG wall as could be expected from

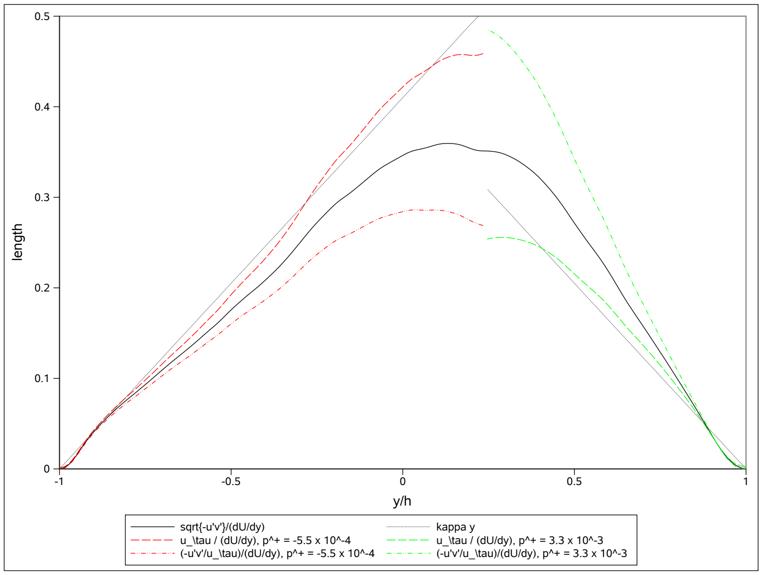

FIGURE 5. Killer Figure.

the smaller friction velocity there. The inflection point may serve as a notional border between zones of influence.

The profiles in wall units and logarithmic y axis in figure 3 reveal a "visual" log layer to y^+ of about 800 on the FPG side, well beyond the centerline. The shear stress has fallen by 40%; this is consistent with Poiseuille and Ekman results. On the APG side, the visual log layer is much shorter, roughly to 280, if it exists at all; opinions will differ, and an exact match is not expected. This is partly due to the lower friction velocity, which lowers y^+ for the same y/h, but as mentioned the pressure gradient is also about 6 times larger in wall units. This layer is much farther from having constant stress than the other one is.

One could conjecture a pressure-gradient law of the wall in which U^+ would be a function of y^+ and p^+, say. Such conjectures have been discussed in the literature, without being accepted; the only durable finding is that the law which would result from the mixing-length assumption (3) is unsuccessful. This cannot be decided on the basis of one or even a few cases. Even in Couette-Poiseuille flow, the same value of p^+ can result from different pairs of Reynolds number and stress ratio, so that this conjecture could be tested with a sufficient set of cases.

Figure 4 shows the production of turbulent kinetic energy, again with an eye to zones of influence. The peak value at the FPG wall is over 10 times larger than at the APG wall, following as expected the fourth power of u_τ (0.3^2 is 0.09). The amount of energy production plausibly attributed to each of the layers is in a ratio closer to 6, which still indicates the dominance of the FPG wall. However, only a small amount of the energy produced near one wall, with small length scales, could "travel" to the other wall.

Figure 5 displays the direct test of (1,2,3) which was the purpose of the study. The three quantities which equal κy in the ideal situation are shown. The buffer layers are thin enough not to invade the regions of interest. The curves for (1) and (2) are discontinuous by a factor $\sqrt{0.3}$, because they involve the friction velocity of the wall they are associated with. The switch is placed at $y = 0.2$, which appears to separate the zones of influence of the two walls, reminiscent of the inflection point in the velocity profile.

The indications in figure 5 are strong: (1) is much closer to being satisfied than (3), which in turn is closer than (2). The logarithmic behaviour extends much farther from the FPG wall than from the APG wall, in y/h units, and a fortiori in y^+ units; this was seen in earlier figures. This is likely linked to the much smaller value of $|p^+|$ on the FPG side, but could also be influenced by our earlier argument of permeation by the region with stronger turbulence, or conceivably by the very sign of the pressure gradient. On the APG side, it will again be debated whether a convincing log layer exists, as a sustained disagreement with (1) of the order of 8% exists. However, (1) is a decidely better approximation than (3) and of course (2), and near-linear behaviour applies roughly up to where the stress has increased to 3/2, i.e., where the deviation from ideal is strong. The result is not trivial.

4. Conclusion

Thi initial Couette-Poiseuille study is successful in the sense of providing a strong preference among the three hypotheses (1,2,3), and one that is the same on the favorable- and adverse-pressure-gradient sides. The Reynolds number is marginal on the APG side, but a decisive increase will have to wait; the present results deserve some attention, and may initiate fruitful reflection and further experimental and DNS studies. The agreement is strong with a neglected set of 1975 articles, which were experimental and had some debatable features, but came from a most respected source.

In terms of the prevalent "semi-theory" of turbulence, the findings are probably unfortunate, in that the apparent favourite formula (1) is not the one with the most intuitive appeal, at least to the authors. It is not local, and there is no doubt that it will break down in very strong APG. Millikan overlap arguments can be made and will predict logarithmic behaviour, which we also believe DNS at higher Reynolds number would confirm. However they are asymptotic ($\tau^+ \to 1$), when the interest here is in the regions where τ^+ is not very close to 1; say where its values reach 2/3, or 3/2. In other words, the difficulty is to explain the manifest, if approximate, maintenance of the log law deep into the defect-law region, in the terminology of overlap arguments. This issue may not be truly a theoretical one, but its practical importance to turbulence modelling is paramount.

Another unfortunate fact is that the mixing-length equation (3) has been used very widely in algebraic turbulence models (recall Johnson & Coakley 1990). It is the most convenient and might be the most plausible, being local. Recent transport-equation models, which dominate CFD, were not designed with the present controversy in mind; they will be tested in a separate study (not that enforcing a different behaviour on them would be a simple matter). They do not directly enforce the mixing-length or eddy-viscosity formulas, which gives them a chance of approaching (1). Cases with weaker pressure gradients, to moderate the adverse pressure gradient from its current value $p^+ = 0.034$, will be conducted to confirm the APG trend, and opportunities towards higher Reynolds number exploited in due time. A case with vanishing skin friction at the APG wall would have a theoretical interest of its own.

Acknowledgments

For what?

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